Higher-order gravity and the cosmological background of gravitational waves

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The cosmological background of gravitational waves can be tuned by the higher-order corrections to the gravitational Lagrangian. In particular, it can be shown that assuming $R^{1+\epsilon}$, where ϵ indicates a generic (eventually small) correction to the Hilbert-Einstein action in the Ricci scalar R, gives a parametric approach to control the evolution and the production mechanism of gravitational waves in the early Universe.

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Several issues coming from Cosmology and Quantum Field Theory suggest to extend the Einstein General Relativity in order to cure shortcomings emerging from observations and self-consistent unification theories. In early time Cosmology, the presence of Big Bang singularity, flatness and horizon problems [1] led to the result that Standard Cosmological Model [2], is inadequate to describe the Universe at extreme regimes. On the other hand, General Relativity is a *classical* theory which does not work as a fundamental theory, when one wants to achieve a full quantum description of spacetime (and then of gravity). Due to these facts and, first of all, to the lack of a self-consistent Quantum Gravity theory, alternative theories of gravity have been pursued in order to attempt, at least, a semi-classical scheme where General Relativity and its positive results could be recovered. A fruitful approach has been that of *Extended Theories of Gravity* which have become a sort of paradigm in the study of gravitational interaction based on corrections and enlargements of the Einstein scheme [3, 4].

Besides fundamental physics motivations, these theories have acquired a huge interest in cosmology due to the fact that they "naturally" exhibit inflationary behaviors able to overcome the shortcomings of Standard Cosmological Model (based on General Relativity). The related cosmological models seem very realistic and, several times, capable of matching with the observations [5, 6]. Recently, Extended Theories of Gravity are going to play an interesting role to describe the today observed Universe. In fact, the amount of good quality data of last decade has made it possible to shed new light on the effective picture of the Universe. Type Ia Supernovae (SNeIa) [7], anisotropies in the cosmic microwave background radiation (CMBR) [8], and matter power spectrum inferred from large galaxy surveys [9] represent the strongest evidences for a radical revision of the Cosmological Standard Model also at recent epochs. In particular, the *concordance* Λ*CDM model* predicts that baryons contribute only for $\sim 4\%$ of the total matter - energy budget, while the exotic *cold dark matter* (CDM) represents the bulk of the matter content ($\sim 25\%$) and the cosmological constant Λ plays the role of the so called "dark energy" ($\sim 70\%$) [10]. Although being the best fit to a wide range of data [11], the ΛCDM model is severely affected by strong theoretical shortcomings [12] that have motivated the search for alternative models [13]. Dark energy models mainly rely on the implicit assumption that Einstein's General Relativity is the correct theory of gravity indeed. Nevertheless, its validity on the larger astrophysical and cosmological scales has never been tested [14], and it is therefore conceivable that both cosmic speed up and dark matter represent signals of a breakdown in our understanding of gravitation law so that one should consider the possibility that the Hilbert - Einstein Lagrangian, linear in the Ricci scalar R, should be generalized [15, 16, 17, 18, 19].

From a genuine astrophysical viewpoint, Extended Theories of Gravity do not urgently require to find out candidates for dark energy and dark matter at fundamental level (till now they have not been found!); the approach starts from taking into account only the "observed" ingredients (i.e. gravity, radiation and baryonic matter); it is in full agreement with the early spirit of General Relativity which could not act in the same way at all scales (for comprehensive reviews on the argument, see [20, 21]). In fact, General Relativity has been successfully probed in the weak filed limit (e.g. Solar System experiments) and also in this case there is room for alternative theories of gravity which are not at all ruled out, as discussed in several recent studies [22, 23, 24, 25]. In particular, it is possible to show that several $f(R)$ models could satisfy both cosmological and Solar System tests [26], could be constrained as the scalar-tensor theories [22, 27], could give rise to new effects capable of explaining the so called Pioneer anomaly (see for example [28] and the references therein).

However, also considering the recent interest in the argument, a comprehensive effective theory of gravity, acting

consistently at any scale, is far, up to now, to be found out, and this demands an improvement of observational datasets and the search for experimentally testable theories. A more pragmatic point of view could be to "reconstruct" the suitable theory of gravity starting from data. The main issues of this "inverse " approach is matching consistently observations at different scales and taking into account wide classes of gravitational theories where "ad hoc" hypotheses are avoided. In principle, the most popular dark energy models can be achieved by considering $f(R)$ theories of gravity [29, 30] and the same track can be followed, at completely different scales, to match galactic dynamics [31]. This philosophy can be taken into account also for the cosmological stochastic background of gravitational waves (GW) which, together whit cosmic microwave background radiation (CMBR) [32], would carry, if detected, a huge amount of information on the early stages of the Universe evolution. However, also in this case, a key role for the production and the detection of this relic graviton background is played by the adopted theory of gravity which gives rise to specific early Universe models [33, 34, 35].

In this letter we want to face the problem of how a theory of gravity with a Lagrangian of the form $R^{1+\epsilon}$, where ϵ is a deviation (e.g. small) with respect to General Relativity, could be related to the cosmological background of GWs. They are perturbations $h_{\mu\nu}$ of the metric $g_{\mu\nu}$ which transform as 3-tensors. Following [2], the GW-equations in the transverse-traceless gauge are

$$\Box h_i^j = 0 \tag{1}$$

where $\Box \equiv (-g)^{-1/2}\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu$ is the d'Alembert operator; these equations are deduced from the weak-field limit, in vacuum, of the Einstein field equations. The Latin indexes run from 1 to 3, the Greek ones from 0 to 3. Our task is now to derive the analog of Eqs. (1) assuming a correction to the Hilbert-Einstein action given by

$$\mathcal{A} = \frac{1}{2k}\int d^4x\sqrt{-g}f_0 R^{1+\epsilon}\,. \tag{2}$$

From now on we will assume units for which $f_0 = 1$. It is easy to show that

$$R^{1+\epsilon} = R\cdot R^{\epsilon} \simeq R\left(1 + (\log R)\epsilon + \mathcal{O}(\epsilon^2)\right) \simeq R + \epsilon R\log R\,, \tag{3}$$

and this form of the action can be assumed: $i)$ to define the right physical dimensions of the coupling constant; $ii)$ to control the magnitude of the corrections with respect to the standard Einstein gravity. Assuming a conformal transformation, the extra degrees of freedom related to the higher order gravity can be recast into a physically significant scalar field (for a detailed discussion see [36, 37] and references therein)

$$\widetilde{g}_{\mu\nu} = e^{2\Phi}g_{\mu\nu} \qquad \text{with} \qquad e^{2\Phi} = \frac{df(R)}{dR} = f'(R) \simeq (1+\epsilon+\epsilon\log R) \tag{4}$$

where the prime indicates the derivative with respect to the Ricci scalar R and Φ is the "conformal scalar field". The conformally equivalent Hilbert-Einstein action is

$$\widetilde{\mathcal{A}} = \frac{1}{2k}\int\sqrt{-\widetilde{g}}d^4x\left[\widetilde{R} + \mathcal{L}\left(\Phi,\Phi_{;\mu}\right)\right] \tag{5}$$

where $\mathcal{L}\left(\Phi,\Phi_{;\mu}\right)$ is the conformal scalar field Lagrangian derived from

$$\widetilde{R} = e^{-2\Phi}\left(R - 6\Box\Phi - 6\Phi_{;\delta}\Phi^{;\delta}\right)\,. \tag{6}$$

Deriving the field equations from (5), the GW-equations are

$$\widetilde{\Box}\tilde{h}_i^j = 0 \tag{7}$$

expressed in the conformal metric $\widetilde{g}_{\mu\nu}$. Since no scalar perturbation couples to the tensor part of gravitational waves, we have

$$\widetilde{h}_i^j = \widetilde{g}^{lj}\delta\widetilde{g}_{il} = e^{-2\Phi}g^{lj}e^{2\Phi}\delta g_{il} = h_i^j \tag{8}$$

which means that h_i^j is a conformal invariant.

As a consequence, the plane-wave amplitudes $h_i^j(t) = h(t)e_i^j\exp(ik_m x^m)$, where e_i^j is the polarization tensor, are the same in both metrics. This fact will assume a key role in the following discussion. The d'Alembert operator transforms as

$$\widetilde{\Box} = e^{-2\Phi}\left(\Box + 2\Phi^{;\lambda}\partial_{;\lambda}\right) \tag{9}$$

and this means that the background is changing with frame while the tensor plane-wave amplitudes not.

In order to study the gravitational waves in the cosmological background, the operator (9) has to be specified for a Friedmann-Robertson-Walker metric and then Eq. (7) becomes

$$\ddot{h}+\left(3H+2\dot{\Phi}\right)\dot{h}+k^2a^{-2}h=0 \tag{10}$$

being $a(t)$ the scale factor, k the wave number and h depending only on the surviving component of h_i^j in the homogeneous and isotropic metric.

It is worth stressing that Eq. (10) applies to any $f(R)$ theory whose conformal transformation can be defined as $e^{2\Phi}=f'(R)$. The GW amplitude depends on the specific cosmological background $a(t)$ and the specific theory of gravity which can be parameterized by ϵ in our case (2). For example, assuming time power law behaviors for $a(t)$ and $f'(R(t))$, we have

$$f'(R)=f'_0\,(t/t_0)^m\,,\;\; a(t)=a_0\,(t/t_0)^n\,. \tag{11}$$

For a generic $f(R)=R^{1+\epsilon}$, it is $\epsilon=-\frac{m}{2}$. Eq. (10) can be recast in the form

$$\ddot{h}+(3n-2\epsilon)\,t^{-1}\dot{h}+k^2a_0\,(t_0/t))^{2n}\,h=0 \tag{12}$$

whose general solutions are J_α's ar Bessel functions. In Fig. (1) some examples are given. The plots are labelled by the set of parameters $\{m,\,n,\,\epsilon\}$ which assign the time evolution of $\Phi(t)$ and $a(t)$ with respect to a given power-law theory.

The time units are in terms of the Hubble radius H^{-1}. It is clear that the conformally invariant plane-wave amplitude evolution of the tensor GW strictly depends on the cosmological background "tuned" by ϵ.

Let us now take into account the issue of the production of GWs contributing to the cosmological stochastic background. Several mechanisms can be considered as vacuum fluctuations, phase transitions [33], cosmological populations of astrophysical sources [38] and so on. In principle, we could seek for contributions due to every high-energy process in the early phases of the Universe evolution.

It is important to distinguish processes coming from transitions like inflation, where the Hubble flow emerges in the radiation dominated phase and processes, like the early star formation rates, where the production takes place during the dust dominated era. In the first case, stochastic GW background is strictly related to the cosmological model. This is the case we are considering here which is strictly connected to the specific theory of gravity. In particular, one can assume that the main contribution to the stochastic background comes from the amplification of vacuum fluctuations at the transition between an inflationary phase and the radiation dominated era. However, in any inflationary model, we can assume that the GWs generated as zero-point fluctuations during the inflation undergo adiabatically damped oscillations ($\sim 1/a$) until they reach the Hubble radius H^{-1}. This is the particle horizon for the growth of perturbations. On the other hand, any previous fluctuation is smoothed away by the inflationary expansion. The GWs freeze out for $a/k \gg H^{-1}$ and reenter the H^{-1} radius after the reheating in the Friedmann era (see also [39, 40]). The reenter in the radiation-dominated or in the dust-dominated era depends on the scale of the GW. After the reenter, GWs can be detected by their Sachs-Wolfe effect on the temperature anisotropy $\triangle T/T$ at the decoupling [41]. If Φ acts as the inflaton we have $\dot{\Phi}\ll H$ during the inflation. Adopting the conformal time $d\eta=dt/a$, Eq. (10) reads

$$h''+2\frac{\chi'}{\chi}h'+k^2h=0 \tag{13}$$

where $\chi=ae^{\Phi}$. The derivation is now with respect to η. An inflationary behavior is achieved for $a(t)=a_0\exp(Ht)$, then $\eta=\int dt/a=(aH)^{-1}$ and $\chi'/\chi=-\eta^{-1}$. The exact solution of (13) is

$$h(\eta)=k^{-3/2}\sqrt{2/k}\,[C_1\,(\sin k\eta-\cos k\eta)+C_2\,(\sin k\eta+\cos k\eta)] \tag{14}$$

Inside the radius H^{-1}, we have $k\eta\gg 1$. Considering the absence of gravitons in the initial vacuum state, we have only negative-frequency modes and then the adiabatic behavior is

$$h=k^{1/2}\sqrt{2/\pi}\frac{1}{aH}C\exp(-ik\eta)\,. \tag{15}$$

where C is the parameter to be tuned later. At the first horizon crossing ($aH=k$), the averaged amplitude $A_h=(k/2\pi)^{3/2}\,|h|$ of the perturbation is

$$A_h=\frac{1}{2\pi^2}C\,. \tag{16}$$

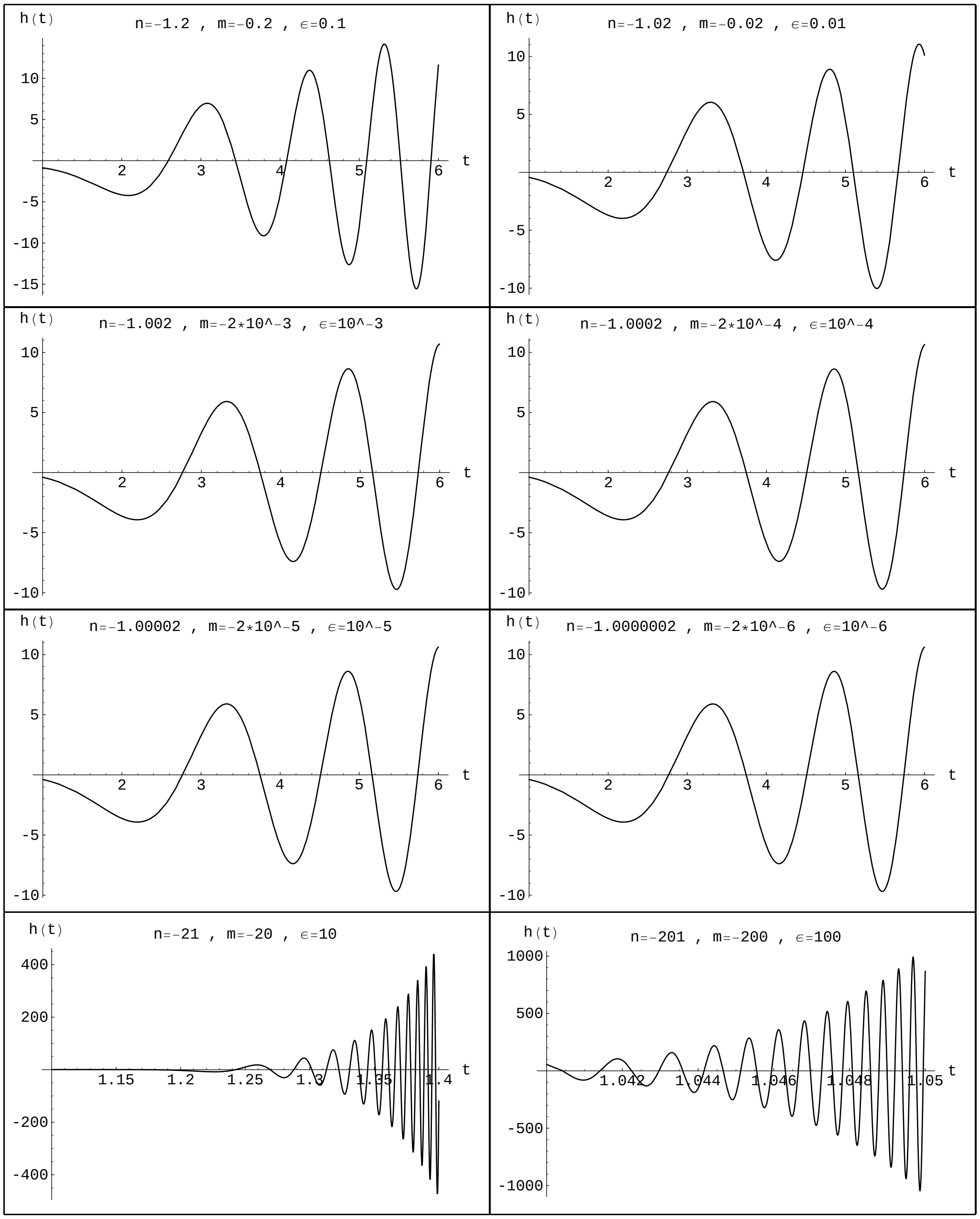

Figure 1: Qualitative evolution of the GW amplitude for some choices of $a(t)$, $\Phi(t)$ and $R^{1+\epsilon}$. Time-scales, amplitudes and "chirps" of GWs strictly depend on the value of ϵ as it is especially shown in the last two (unrealistic) cases (ϵ extremely large).

When the scale a/k becomes larger than the Hubble radius H^{-1}, the growing mode of evolution is constant, i.e. it is frozen. This situation corresponds to the limit $-k\eta \ll 1$ in Eq. (14). Since Φ acts as the inflaton field, we have $\Phi \sim 0$ at reenter (after the end of inflation). Then the amplitude A_h of the wave is preserved until the second horizon crossing; after it can be observed, in principle, as an anisotropy perturbation on the CMBR. It can be shown that $\triangle T/T \lesssim A_h$, as an upper limit to A_h, since other effects can contribute to the background anisotropy. From this consideration, it is clear that the only relevant quantity is the initial amplitude C in Eq. (15), which is conserved until the reenter. Such an amplitude depends on the fundamental mechanism generating perturbations. Inflation gives rise to processes capable of producing perturbations as zero-point energy fluctuations. Such a mechanism depends on the gravitational interaction and then $(\triangle T/T)$ could constitute a further constraint to select a suitable theory of gravity. Considering a single graviton in the form of a monochromatic wave, its zero-point amplitude is derived through the

commutation relations:

$$[h(t,x),\, \pi_h(t,y)] = i\delta^3(x-y) \tag{17}$$

calculated at a fixed time t, where the amplitude h is the field and π_h is the conjugate momentum operator. Writing the Lagrangian for h

$$\widetilde{\mathcal{L}} = \frac{1}{2}\sqrt{-\widetilde{g}}\widetilde{g}^{\mu\nu}h_{;\mu}h_{;\nu} \tag{18}$$

in the conformal FRW metric $\widetilde{g}_{\mu\nu}$ (recall that the amplitude h is conformally invariant), we obtain

$$\pi_h = \frac{\partial\widetilde{\mathcal{L}}}{\partial\dot{h}} = e^{2\Phi}a^3\dot{h} \tag{19}$$

Eq. (17) becomes

$$\left[h(t,x),\, \dot{h}(y,y)\right] = i\frac{\delta^3(x-y)}{a^3e^{2\Phi}} \tag{20}$$

and the fields h and $\dot{h}$ can be expanded in terms of creation and annihilation operators

$$h(t,x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\left[h(t)e^{-ikx} + h^*(t)e^{+ikx}\right], \tag{21}$$

$$\dot{h}(t,x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\left[\dot{h}(t)e^{-ikx} + \dot{h}^*(t)e^{+ikx}\right]. \tag{22}$$

The commutation relations in conformal time are then

$$[hh'^* - h^*h'] = \frac{i(2\pi)^3}{a^3e^{2\Phi}} \tag{23}$$

Using (15) and (16), we obtain $C = \sqrt{2}\pi^2He^{-\Phi}$, where H and Φ are calculated at the first horizon-crossing and

$$A_h = \frac{\sqrt{2}}{2}He^{-\Phi} \simeq \frac{\sqrt{2}}{2}H\left(1 - \Phi + \frac{1}{2}\Phi^2\cdots\right) = \frac{\sqrt{2}}{2}H\left(1 - \frac{\epsilon}{2}\log R + \frac{\epsilon^2}{8}(\log R)^2\cdots\right)\ , \tag{24}$$

or, in general,

$$A_h = \frac{H}{\sqrt{2f'(R)}}\,. \tag{25}$$

Clearly the amplitude of GWs produced during inflation depends on the given theory of gravity that, if different from General Relativity, gives extra degrees of freedom capable of affecting the cosmological dynamics. On the other hand, the Sachs-Wolfe effect related to the CMBR temperature anisotropy could constitute a test for the theory of gravity at early epochs, i.e. at very high redshift. This probe could give further constraints on the GW-stochastic background, if Extended Theories of Gravity are independently probed at other scales. In summary, assuming that primordial vacuum fluctuations produce GWs, beside scalar perturbations, kinematical distortions and so on, the initial amplitude of these ones is a function of the assumed theory of gravity and then the stochastic background can, in a certain sense, be "tuned" by the theory. Conversely, data coming from the Sachs-Wolfe effect could contribute to select a suitable ϵ which can be consistently matched with other observations. If eventually, one finds ϵ rigorously zero, this could be a further test for General Relativity also at very early cosmological scales. This result could be obtained very soon through space and ground based interferometers [42].

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